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Sign-Stability Concept of Ecology for Control Design with Aerospace Applications

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The main aim of this research is to understand the underlying features of natural systems like Eco/Bio systems, which tend to be highly robust under perturbations, and then apply these principles to build a robust engineering system. Toward this direction, some fundamental qualitative features of ecological sign stability are reviewed and transformed into a set of mathematical results in matrix theory with quantitative information, which is usually encountered in engineering sciences. In particular, the effect of the signs of elements of a matrix on the matrix properties such as eigenvalues and condition number is shown. Similarly, it is also shown that under some assumptions on the magnitudes of the elements, predator–prey phenomena render some special properties like normality to matrices. These properties in turn are shown to impart superior robustness bounds for a class of sign-stable matrices. Then the issue of controller design is addressed, and efforts are made to identify target closed-loop systems that incorporate the desirable features of ecological systems. For a closed-loop system satisfying these properties, an algorithm for the design of controller is given. This control design procedure is illustrated with the help of two applications in the Aerospace field: satellite attitude control and aircraft lateral dynamics control. The results presented in this paper can assist in the use of ecological system principles to build highly robust engineering systems.

I. Introduction

IT IS well-recognized that natural systems such as ecological and biological systems are highly robust under various perturbations. On the other hand, engineered systems can be made highly optimal for good performance, but they tend to be nonrobust under perturbations. Thus, it is natural and essential for engineers to delve into the question of what the underlying features of natural systems are, what makes them so robust, and then try to apply these principles to make the engineered systems more robust. The research reported in this paper is an attempt to make a contribution in this aspect. Toward this objective, the interesting aspect of qualitative stability in ecological systems is considered in particular. The fields of population biology and ecology deal with the analysis of growth and decline of populations in nature and the struggle of species to predominate over one another. The existence or extinction of a species, apart from its own effect, depends on its interactions with various other species in the ecosystem it belongs to. Hence, the type of interaction is very critical to the sustenance of species. This paper attempts to study these interactions and their nature thoroughly and investigate the effect of these qualitative interactions on the quantitative properties of matrices, specifically on three matrix properties, namely, eigenvalue distribution, normality/condition number, and robust-stability analysis. This type of study is important for researchers in both fields, because qualitative properties do have significant impact on the quantitative aspects. This paper attempts to bring out that interrelationship in a sound mathematical framework. In addition, these properties are in turn exploited in the design of controllers for engineering systems to make them more robust. With this backdrop the paper is organized as follows: in the next section, the basic principles of ecosystem interactions are briefly reviewed. In Section III, the concept of ecological sign stability for an ecosystem is analyzed in detail, and new results on the matrix-theory implications of the effect of those interactions on three specific characteristics, namely eigenvalue distribution, normality/ condition number, and robust stability are presented. The relevance of these ecology principles to engineering systems is highlighted by showing that a special class of ecological sign-stable matrices is more robust than the same class of standard Hurwitz-stable matrices. The proposed theory is illustrated with various examples. Aided by this result, in Section IV, the design of a controller using ideas from ecological sign stability is presented. The proposed controller-design procedure is illustrated with the help of two application examples in the aerospace field: one for a satellite attitude control problem and another for an aircraft lateral dynamics problem. Finally, concluding remarks are offered in Section V.

II. Brief Review of Ecological Principles

In this section a few ecological system principles that are relevant to this research are briefly reviewed. In a complex community composed of many species, numerous interactions take place. These interactions in ecosystems can be broadly classified as 1) mutualism, 2) commensalism/ammensalism, 3) competition, and 4) predation (parasitism). Mutualism occurs when both species benefit from the interaction. When one species benefits/suffers and the other one remains unaffected, the interaction is classified as commensalism/ amensalism. When species compete with each other, that interaction is known as competition. Finally, if one species is benefited and the other suffers, the interaction is known as predation (parasitism). In ecology, the magnitudes of the mutual effects of species on each other are seldom precisely known, but one can establish with certainty the types of interactions that are present. Many mathematical population models were proposed over the last few decades to study the dynamics of eco/bio systems, which are discussed in textbooks [1,2]. The most significant contributions in this area come from the works of Lotka and Volterra. The following is a model of a predator–prey interaction where x is the prey and y is the predator:

$$\dot{x} = xf(x, y) \qquad \dot{y} = yg(x, y) \tag{1}$$

where it is assumed that $\partial f(x, y)/\partial y < 0$ and $\partial g(x, y)/\partial x > 0$. This means that the effect of y on the rate of change of $x(\dot{x})$ is negative, while the effect of x on the rate of change of $y(\dot{y})$ is positive. A simple example is

$$\dot{R}_1 = R_1(b - pR_2) \quad \text{(prey)} \tag{2a}$$

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$$\dot{R}_2 = R_2(rR_1 - d) \quad \text{(predator)} \tag{2b}$$

where R_1 is prey density, R_2 is predator density, b is the intrinsic rate of prey-population increase, p is predation-rate coefficient, r is the reproduction rate of predators per one prey eaten, and d is the predator mortality rate, and all these are positive parameters. The predation–prey nature can be clearly seen in this simplified model. The negative coefficient of the R_1R_2 term in the equation for $\dot{R}_1$ and the positive coefficient of the R_1R_2 term in the equation for $\dot{R}_2$ are the manifestation of the predator–prey interaction between the species.

A more general n-species Lotka–Volterra system can be represented as [3,4]:

$$\frac{dR_i}{dt} = R_i\left[b_i + \sum_{j=1}^{n} a_{ij}R_j\right], \qquad i = 1, 2, \cdots, n \tag{3}$$

where no mathematical constraints are imposed on the various coefficients. These equations may represent predator–prey, mutualism, or competition cases. The predator–prey category of Lotka–Volterra models has mathematical constraints on the coefficients such that there is mutual negative and positive effect of n species on each other. Such models are computationally challenging, and, hence, finding better numerical solutions and obtaining approximate analytical solutions of these systems [5] is a topic for continued research. Though originally formulated to describe the time history of a biological system, this model finds its application in a number of engineering fields [6,7]. In fact, the one-predator and one-prey model is one of the most popular ones used to demonstrate a simple nonlinear control-system model. Numerous other models of predator–prey systems were also proposed in the literature [1,2]. These models include many parameters that make the model more real and more difficult to analyze. These also include models of increasing levels of complexity, which account for continually acting unpredictable disturbances like migrating species, diseases, change in climatic conditions, etc., which in turn decrease the accuracy of the predictions. Noting that various predator–prey models are extensively studied in the field of engineering, in this current research and in this paper, for brevity in exposition, this study is restricted to Lotka–Volterra models and their various properties. It is of interest to note that there is considerable research being carried out on non-predator–prey-type models such as compartmental models as in [8–11], which attest to the growing interest in the interrelationship between life-sciences research and engineering-sciences research.

The stability of the equilibrium solutions of these models has been a subject of intense study in life sciences [12]. These models and the stability of such systems give deep insight into the balance in nature. If a state of equilibrium can be determined for an ecosystem, it becomes inevitable to study the effect of perturbation of any kind in the population of the species on the equilibrium. These small perturbations from equilibrium can be modeled as linear state-space systems, where the state-space plant matrix is the Jacobian. This means that, technically, in the Jacobian matrix, one does not know the actual magnitudes of the partial derivatives, but their signs are known with certainty. That is, the nature of the interaction is known but not the intensity levels of those interactions. As mentioned before, there are four classes of interactions, and after linearization they can be represented in the following manner.

In Table 1, column two is a visual representation of such interactions and is known as a directed graph, or digraph [13], while column three is the matrix representation of the interaction between two species. * represents the effect of a species on itself.

In other words, in the Jacobian matrix, the qualitative information about the species is represented by the signs $+$, $-$, or 0. Thus, the (i, j)th entry of the state-space (Jacobian) matrix simply consists of signs $+$, $-$, or 0, with the $+$ sign indicating species j having a positive influence on species i, the $-$ sign indicating negative influence, and 0 indicating no influence. The diagonal elements give information regarding the effect of a species on itself. Negative means the species is self-regulatory, positive means it aids the growth of its own population, and zero means it has no effect on itself. For example, in Fig. 1 below, each of the sign-pattern matrices A_1, A_2, and A_3 are the Jacobian form, while D_1, D_2, and D_3 are their corresponding digraphs, A_1 being an ecosystem with three species and A_2 and A_3 being ecosystems with five and six species, respectively.

The question then is whether it can be concluded, just from this sign pattern, whether the system is stable or not. If so, the system is said to be qualitatively stable. In some literature, this concept is also labeled as *sign stability*. In what follows, these two terms are used interchangeably. It is important to keep in mind that the systems (matrices) that are qualitatively stable (sign stable) are also stable in the ordinary sense. That is, qualitative stability implies Hurwitz

$$A_1 = \begin{bmatrix} 0 & + & + \\ - & 0 & 0 \\ - & 0 & - \end{bmatrix} \qquad D_1:$$

a)

$$A_2 = \begin{bmatrix} 0 & - & 0 & 0 & 0 \\ + & 0 & - & 0 & 0 \\ 0 & + & - & - & 0 \\ 0 & 0 & + & 0 & - \\ 0 & 0 & 0 & + & 0 \end{bmatrix} \qquad D_2:$$

b)

$$A_3 = \begin{bmatrix} 0 & 0 & 0 & + & 0 & 0 \\ 0 & 0 & 0 & 0 & + & 0 \\ 0 & 0 & 0 & 0 & 0 & + \\ - & 0 & 0 & - & 0 & + \\ 0 & - & 0 & + & - & 0 \\ 0 & 0 & - & 0 & + & - \end{bmatrix} \qquad D_3:$$

c)

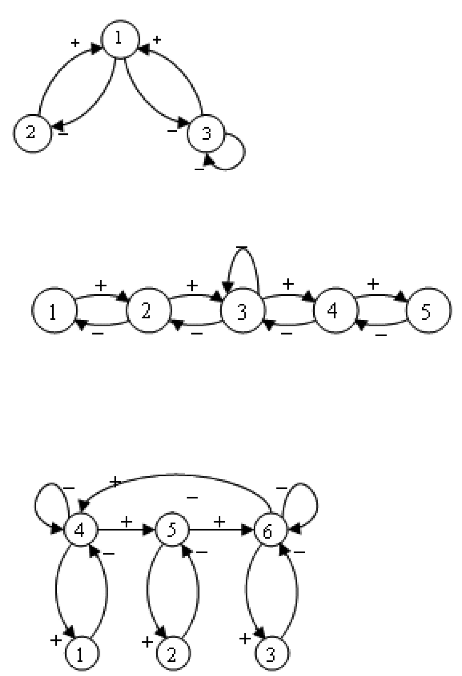


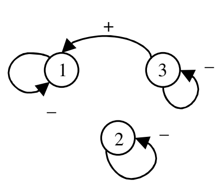
Fig. 1 Various sign patterns and their corresponding digraphs representing ecological systems: a) three-species system, b) five-species system, and c) six-species system.

stability (eigenvalues with negative real part) in the ordinary sense of engineering sciences. In other words, once a particular sign matrix is shown to be qualitatively (sign) stable, any magnitude can be inserted in those entries, and for all those magnitudes the matrix is automatically Hurwitz stable. This is the most attractive feature of a sign-stable matrix. However, the converse is not true. Systems that are not qualitatively stable can still be stable in the ordinary sense for certain appropriate magnitudes in the entries. From now on, to distinguish from the concept of qualitative stability of life-sciences literature, the label of *quantitative stability* for the standard Hurwitz stability in engineering sciences is used.

Kaszkurewicz and Bhaya [14] briefly discuss sign stability in the context of matrix diagonal stability in systems and computation, and they provide a few references within the book. However, these references touch upon the sufficient conditions for sign stability and do not allude to the color-test conditions, which are part of the necessary and sufficient conditions provided in the ecology literature. In [15–17], necessary and sufficient conditions for qualitative stability of an ecosystem were given. These ecological-sign-stability conditions, stated in terms of ecology, were interpreted in matrix-theory notation, and these conditions were transformed into an algorithm to test the sign stability of a given sign matrix in [18,19]. With this algorithm, all matrices that are sign stable can be stored a priori as discussed in [20]. If a sign pattern in a given matrix satisfies the conditions given in the above papers (thus in the algorithm), it is an ecological sign-stable pattern, and, hence, that matrix is Hurwitz stable for any magnitudes in its entries. A subtle distinction between sign-stable matrices and ecological sign-stable matrices is now made, emphasizing the role of nature in interactions. Though the property of Hurwitz stability is held in both cases, ecosystems sustain solely because of interactions between various species. In matrix notation this means that the nature of off-diagonal elements is essential for an ecosystem. Consider a strictly upper-triangular 3×3 matrix:

$$A = \begin{bmatrix} - & 0 & + \\ 0 & - & 0 \\ 0 & 0 & - \end{bmatrix}$$

From the quantitative viewpoint, it is seen that the matrix is Hurwitz stable for any magnitudes in the entries of the matrix. This means that it is indeed (qualitatively) sign stable. But its digraph has the following structure:



Since there is no predator–prey link and in fact no link at all between species 1 and 2 and 3 and 2, such a digraph cannot represent an ecosystem. Therefore, though a matrix is sign stable, it need not belong to the class of ecological sign-stable matrices. In Fig. 2, these various classes of sign patterns and the corresponding relationship between these classes is depicted. So, every ecological sign-stable pattern is sign stable, but the converse is not true.

With this brief review of ecological system principles, the implications of these ecological qualitative principles on three quantitative matrix-theory properties (namely eigenvalues, normality/condition number, and robust stability) are investigated. In particular, in the next section, new results that clearly establish these implications are presented. As mentioned in the previous section, the motivation for this study and analysis is to exploit some of these desirable features of ecological system principles to design controllers for engineering systems to make them more robust.

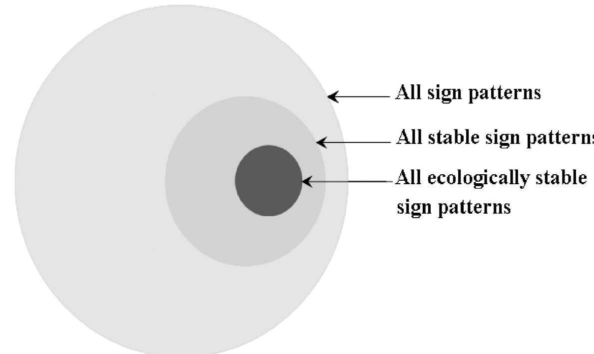


Fig. 2 Classification of sign patterns.

III. Ecological Sign Stability and its Implications in Quantitative Matrix Theory

In this major section of the paper, the implications of the ecological-sign-stability aspect discussed above on the quantitative matrix theory are established. In particular, the section offers three explicit contributions to expand the current knowledge base: 1) eigenvalue distribution of sign-stable matrices, 2) normality/condition-number properties of ecological sign-stable matrices, and 3) robustness properties of ecological sign-stable matrices. These three contributions in turn help in determining the role of magnitudes in quantitative ecological sign-stable matrices. This type of information is clearly helpful in designing robust controllers as shown in later sections. With this motivation, a three-species ecosystem is thoroughly analyzed, and the ecological principles (in terms of matrix properties that are of interest in engineering systems) are interpreted. This section is organized as follows: first, new results on the eigenvalue distribution of ecological sign-stable matrices are presented; second, considering ecological systems with only predation–prey-type interactions, it is shown how selection of appropriate magnitudes in these interactions imparts the property of normality (and thus highly desirable condition numbers) in matrices. In what follows, for each of these cases, concepts are first discussed from an ecological perspective, and then the resulting matrix theory implications from a quantitative perspective are presented.

A. Stability and Eigenvalue Distribution

Stability is the most fundamental property of interest to all dynamic systems. Clearly, in time-invariant matrix theory, stability of matrices is governed by the negative real part nature of its eigenvalues. It is always useful to get bounds on the eigenvalue distribution of a matrix with as little computation as possible, it is hoped as directly as possible from the elements of that matrix. It turns out that sign-stable matrices have interesting eigenvalue distribution bounds. A few new results are now presented in this aspect.

1. Ecological Perspective: Nature of Interactions and Their Role in Stability

In a qualitative stable ecosystem, two species cannot have positive effect on themselves or on each other, because this means that their population might grow without bound. Hence the case of mutualism is eliminated. A stable system also eliminates the case of competition, which amounts to extinction. Therefore, a stable ecosystem is essentially made of self-regulatory (or no regulation at least) species that have only predator–prey/parasitic and commensal/ammensal interactions.

2. Quantitative Engineering Perspective: Eigenvalue Distribution of Sign-Stable Matrices

In what follows, the quantitative matrix-theory properties for an n-species ecological system is established; i.e., an $n \times n$ sign-stable matrix with predator–prey and commensal/ammensal interactions is considered, and its eigenvalue distribution is analyzed. In particular, various cases of diagonal elements' nature, which are shown to possess some interesting eigenvalue distribution properties, are considered.

Theorem 1 (case of all negative diagonal elements): for all $n \times n$ sign-stable matrices, with all negative diagonal elements, the bounds on the real parts of the eigenvalues are given as follows: the lower bound on the magnitude of the real part is given by the minimum-magnitude diagonal element, and the upper bound is given by the maximum-magnitude diagonal element in the matrix. That is, for an $n \times n$ ecological sign-stable matrix $A = [a_{ij}]$,

$$|a_{ii}|_{\min} \leq |\mathrm{Re}(\lambda)|_{\min} \leq |\mathrm{Re}(\lambda)|_{\max} \leq |a_{ii}|_{\max} \tag{4}$$

Proof: the characteristic equation for an $n \times n$, real, Hurwitz-stable matrix A is given by $\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_n = 0$, where $a_1 = -\mathrm{trace}(A)$ and the other coefficient a_i ($i = 1, 2, \cdots, n$) satisfy the positive Hurwitz-determinant (Routh–Hurwitz) conditions. Note that in a sign-stable matrix $a_1 = -\mathrm{trace}(A) = \sum a_{ii} = \sum_{i=1}^{n} -|\mathrm{Re}\lambda_i|$, where $a_{ii} < 0$ for all i.

In a sign-stable matrix, where Hurwitz stability is satisfied independent of the magnitudes of the elements, it is clear that the real parts of the eigenvalues are always negative for any magnitudes in the entries of the matrix. The absolute values of the real parts of the eigenvalues are solely dependent on the magnitudes of the diagonal elements, and the imaginary parts of these eigenvalues are decided by the off-diagonal elements of the sign-stable matrix. Hence,

$$|a_{ii}|_{\min} \leq |\mathrm{Re}(\lambda)|_{\min} \leq |\mathrm{Re}(\lambda)|_{\max} \leq |a_{ii}|_{\max} \tag{5}$$

Example 1: consider the sign-pattern matrix of a 3×3 ecological sign-stable matrix that has one predator–prey link and two ammensal interactions and all self-regulating species, namely the matrix given by

$$A = \begin{bmatrix} - & - & - \\ 0 & - & 0 \\ + & - & - \end{bmatrix}$$

Selecting a random set of numbers (0.5, 1.2, 4) that the elements of the above matrix can take, all possible matrices are formed, each element having the magnitude from this set of numbers. Hence, there are 2187 quantitative sign-stable matrices with the same sign pattern. Noting that the maximum magnitude is 4 and the minimum magnitude is 0.5, the eigenvalue distribution of all the matrices shown in Fig. 3 verifies the validity of the above theorem, with the lower limit on the real part of the eigenvalues being 0.5 and the upper limit being 4.

Corollary (case of some diagonal elements being zero): if the ecological sign-stable matrix has zeros on the diagonal, the bounds are given by

$$|a_{ii}|_{\min}(=0) < |\mathrm{Re}(\lambda)|_{\min} \leq |\mathrm{Re}(\lambda)|_{\max} \leq |a_{ii}|_{\max} \tag{6}$$

Example 2: the sign pattern in Example 1 has all negative diagonal elements. In this example, the case discussed in the corollary, where one of the diagonal elements is zero, is considered. This sign pattern is as shown in the matrix below:

$$A = \begin{bmatrix} - & - & - \\ 0 & - & 0 \\ + & + & 0 \end{bmatrix}$$

Figure 4 clearly shows the validity of the corollary. The same observation holds even when there are two zeros on the diagonal.

It can be seen that these theorems offer significant insight into the eigenvalue distribution of $n \times n$ ecological sign-stable matrices. Note that the bounds can be simply read off from the magnitudes of the elements of the matrices. This is quite in contrast to the general quantitative Hurwitz-stable matrices, where the lower and upper bounds on the eigenvalues of a matrix are given in terms of the

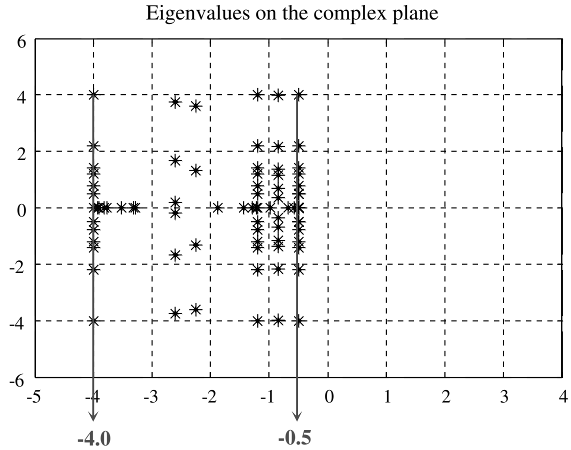


Fig. 3 Eigenvalue distribution of sign-stable matrices with all negative diagonal elements.

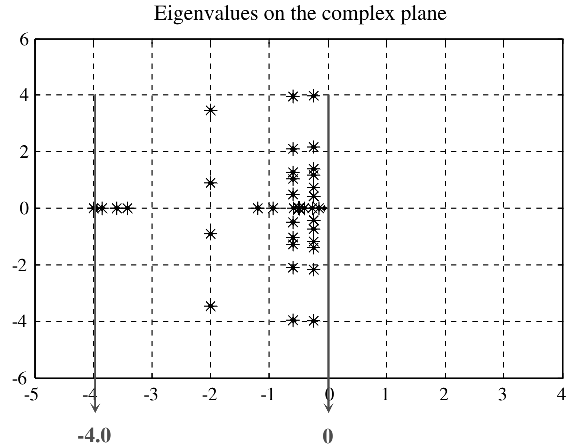


Fig. 4 Eigenvalue distribution of sign-stable matrices with some zero-diagonal elements.

singular values of the matrix and/or the eigenvalues of the symmetric part and skew-symmetric parts of the matrices, using the concept of field of values, which obviously require much computation and are complicated functions of the elements of the matrices.

Now label the ecological sign-stable matrices with magnitudes inserted in the elements as *quantitative ecological sign-stable matrices*. Note that these magnitudes can be arbitrary in each nonzero entry of the matrix. It is interesting and important to realize that these bounds, based solely on sign stability, do not reflect diagonal dominance, which is the typical case with general Hurwitz-stable matrices. Thus it is the diagonal connectivity that is important in these quantitative ecological sign-stable matrices and not the diagonal dominance.

B. Normality and Condition Number

Based on this new insight on the eigenvalue distribution of sign-stable matrices, in what follows, other matrix-theory properties of sign-stable matrices are investigated. In particular, another matrix-theory property of normality/condition number is studied. But this time, the focus is only on ecological sign-stable matrices with pure predator–prey links with no other types of interactions.

1. Ecological Perspective: Role of Pure Predator–Prey Interactions

Apart from the self-regulatory characteristics of species, the phenomena that contribute to the stability of the system are the types of interactions. Since a predator–prey interaction has a regulating effect on both species, predator–prey interactions are of interest in this stability analysis. To study the role played by these interactions, henceforth, the focus is on systems with pure predator–prey links. For a matrix A, pure-predator–prey-link structure implies the following: 1) $A_{ij}.A_{ji} \leq 0 \ \forall \, i, j$ and 2) $A_{ij}.A_{ji} = 0$ if and only if $A_{ij} = A_{ji} = 0$.

Consider three-species systems. For sign-stable three-species ecosystems, since no species can have positive effect on itself, all the diagonal elements are either negative or zero. Hence there can be 1) all negative diagonal elements, 2) one zero and two negative diagonal elements, or 3) two zeros and one negative diagonal element. Based on the predator–prey links, these matrices can be further categorized as ecosystems with the following traits:

1) They have no predator–prey links: $a_{ij} = a_{ji} = 0 \ \forall \, i, j$.

2) They have one predator–prey link: this means there is only one $a_{ij} \times a_{ji} < 0$ situation.

3) They have two predator–prey links: this means there is only one $a_{ij} \times a_{ji} = 0$ situation.

4) There can never be three predator–prey links in a three-species qualitatively stable ecosystem.

Combining the first two of these categories, the set of pure-predator–prey-link matrices can be arranged as follows. A zero diagonal element implies that a species has no control over its growth/decay rate, and so, in order to regulate the population of the species, it is essential that it be connected to at least one predator–prey link. Hence, in Case I, there is need for a minimum of two predator–prey links as there are two zeros on the diagonal (two species have no control on their growth/decay). Similarly, in Case II, since there is one zero on the diagonal, the minimum number of predator–prey links is one. In the case where all diagonal elements are negative, the matrix represents an ecosystem with all self-regulating species. If every species has control over its regulation, a limiting case for stability is a system with no interspecies interactions. This means that there need not be any predator–prey interactions. This is a trivial ecosystem, and such matrices actually belong to the only sign-stable set, not to an ecological sign-stable set.

2. *Quantitative Engineering Perspective: Negative Diagonal Elements and Off-Diagonal Elements of Opposite Sign*

But matrices with all negative diagonal elements are in general desirable from a stability point of view. Ecological sign-stable matrices have predator–prey links, and these types of interactions impart certain desirable features in matrix properties such as the condition number. Hence, in what follows, matrices with all negative diagonal elements and with two predator–prey links are considered.

Consider all 3×3 matrices for Case I-3. Thus, all three diagonal elements have to be negative. There are three off-diagonal element pairs with opposite signs representing pure predation–prey links, namely pairs: $[a_{12}, a_{21}]$, $[a_{13}, a_{31}]$, and $[a_{23}, a_{32}]$. Since in Case I-3, which necessarily admits a maximum of two pure predator–prey links in a 3×3 matrix, one of the pairs above can be taken to be both zero elements. Thus there are 12 ecological sign-stable matrices belonging to Case I-3:

$$\begin{bmatrix} - & 0 & + \\ 0 & - & + \\ - & - & - \end{bmatrix} \begin{bmatrix} - & 0 & - \\ 0 & - & + \\ + & - & - \end{bmatrix} \begin{bmatrix} - & 0 & + \\ 0 & - & - \\ - & + & - \end{bmatrix} \begin{bmatrix} - & 0 & - \\ 0 & - & - \\ + & + & - \end{bmatrix}$$

$$\times \begin{bmatrix} - & - & 0 \\ + & - & + \\ 0 & - & - \end{bmatrix} \begin{bmatrix} - & + & 0 \\ - & - & + \\ 0 & - & - \end{bmatrix} \begin{bmatrix} - & - & 0 \\ + & - & - \\ 0 & + & - \end{bmatrix} \begin{bmatrix} - & + & 0 \\ - & - & - \\ 0 & + & - \end{bmatrix}$$

$$\times \begin{bmatrix} - & + & + \\ - & - & 0 \\ - & 0 & - \end{bmatrix} \begin{bmatrix} - & - & + \\ + & - & 0 \\ - & 0 & - \end{bmatrix} \begin{bmatrix} - & + & - \\ - & - & 0 \\ + & 0 & - \end{bmatrix} \begin{bmatrix} - & - & - \\ + & - & 0 \\ + & 0 & - \end{bmatrix}$$

Now, consider some special quantitative ecological stable matrices of these 12 matrices in the sense that the magnitudes of the elements are selected in a specific way.

Case I: all the diagonal-element magnitudes are equal, and all of the pure predation–prey link element magnitudes are also equal to the magnitude of the diagonal element. That is, the magnitudes of all nonzero elements are equal. As an example, consider

$$\begin{bmatrix} -2 & +2 & 0 \\ -2 & -2 & -2 \\ 0 & +2 & -2 \end{bmatrix}$$

Applying Theorem 1 to this matrix, it is concluded that this matrix has all three eigenvalues with equal (negative) real parts. The matrix is thus is a normal matrix (with the property that $A \times A^T = A^T \times A$) [21]. This in turn implies that the modal matrix of this matrix is orthogonal, resulting in the observation that it has a condition number of one, which is an extremely desirable property for all matrices occurring in engineering applications.

Thus, all of the 12 matrices given above with ones in all of their nonzero entries are indeed normal matrices, and all of them therefore have modal matrices with condition number one.

Case II: in this case, the pure-predator–prey-link element magnitudes are not necessarily equal to the magnitude of the diagonal elements. The intensity of interaction in a single predator–prey link should be equal, but it is not necessary that all the predator–prey links have the same intensity. The matrix below is an example for this case:

$$\begin{bmatrix} -2 & +3 & 0 \\ -3 & -2 & -5 \\ 0 & +5 & -2 \end{bmatrix}$$

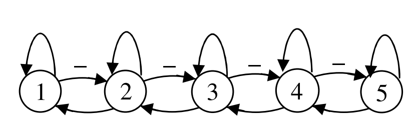
For this case also, the matrix is normal, and the modal matrix has a condition number equal to one.

3. *Generalization to Higher-Order Matrices*

The property of normality is observed in higher order systems too. An ecosystem with purely predator–prey-link interactions is represented by the following digraph for a five-species system. The sign-pattern matrix A represents this digraph.

$$A = \begin{bmatrix} - & + & 0 & 0 & 0 \\ - & - & + & 0 & 0 \\ 0 & - & - & + & 0 \\ 0 & 0 & - & - & + \\ 0 & 0 & 0 & - & - \end{bmatrix}$$

$D:$



For example, if both the strengths of interactions and diagonal elements are unity, the qualitative matrix with maximum number of pure predator–prey links becomes the following quantitative matrix:

$$A = \begin{bmatrix} -1 & +1 & 0 & 0 & 0 \\ -1 & -1 & +1 & 0 & 0 \\ 0 & -1 & -1 & +1 & 0 \\ 0 & 0 & -1 & -1 & +1 \\ 0 & 0 & 0 & -1 & -1 \end{bmatrix}$$

For such a matrix, it can be easily observed that the condition number of its modal matrix is always one. In other words, it is clearly seen that the predator–prey links with equal strengths of interactions (an ecology-derived principle) are imparting the feature of normality (a quantitative feature of matrix theory used in engineering systems).

The example below clearly demonstrates the influence of predator–prey links on the properties of the matrix. This matrix can easily be made a Hurwitz-stable, nonecological sign-stable matrix by changing element (4, 5) from $+1$ to -1:

$$\begin{bmatrix} -1 & +1 & 0 & 0 & 0 \\ -1 & -1 & +1 & 0 & 0 \\ 0 & -1 & -1 & +1 & 0 \\ 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & -1 & -1 \end{bmatrix}$$

For this matrix, the condition number of the modal matrix is 3.0419. Thus it is clear that while the condition number of the modal matrix for a matrix with purely predator–prey links of equal interaction intensities is always one, it is not equal to one once predation–prey property is lost. Therefore, it can be concluded that the existence of predator–prey links in a matrix improves some of its matrix properties, which can be used to improve the performance of a quantitative engineering system. As observed in the case of 3×3 matrices, it is sufficient that the interaction strengths in a predator–prey link be equal. For the condition of normality to hold, all the interaction strengths need not be equal. This means that even

$$\begin{bmatrix} -1 & +1 & 0 & 0 & 0 \\ -1 & -1 & +3 & 0 & 0 \\ 0 & -3 & -1 & +7.5 & 0 \\ 0 & 0 & -7.5 & -1 & +4 \\ 0 & 0 & 0 & -4 & -1 \end{bmatrix}$$

is a normal matrix.

Based on the above observations, the following theorem can be stated and proved:

Theorem 2: an $n \times n$ matrix A with equal diagonal elements and equal predation–prey interaction strengths for each predation–prey link is a normal matrix (Proof given in Appendix).

C. Robust-Stability Properties of Ecological Sign-Stable Matrices

The third and final contribution of this section is related to the connection between ecological sign stability and robust stability in engineering systems.

1. *Ecological Perspective: Independence from Magnitude Information*

As mentioned earlier, the most interesting feature of ecological sign-stable matrices is that the stability property is independent of the magnitude information in the entries of the matrix. Thus the natures of interactions, which in turn decide the signs of the matrix entries and their locations in the matrix, are sufficient to establish the stability of the given sign matrix. Clearly, it is this independence (or nondependence) from magnitude information that imparts the property of robust stability to engineering systems. This aspect of robust stability in engineering systems is elaborated next from a quantitative matrix-theory point of view.

2. *Quantitative Engineering Perspective: Robust Stability of Matrix Families*

In mathematical sciences, the aspect of robust stability of families of matrices has been an active topic of research for many decades. This aspect essentially arises in many applications of system and control theory. When the system is described by linear state-space representation, the plant-matrix elements typically depend on some uncertain parameters, which vary within a given bounded interval.

In the early eighties and nineties, widespread research on robust stability of linear state-space systems with structured real parameter uncertainty was reported in the literature [22,23]. The problem formulation in that research was the question, given a Hurwitz-stable matrix A, of how much perturbation E can be tolerated to maintain the stability of the perturbed matrix $A + E$. When bounds on the norm of E are given to maintain stability, it is labeled as robust stability for unstructured, norm-bounded uncertainty. When bounds on the individual elements of the matrix are important to maintain stability, it is labeled as robust stability for structured real-parameter uncertainty. The interval-matrix problem (or more generally the linear interval-parameter-matrix-family problem, in which the given uncertain parameters vary within a given interval range with a lower and upper bound on the parameters) then became a special case of this structured-uncertainty-problem formulation. Many sufficient conditions were given throughout the literature, which were summarized in [23]. In this area, it was extremely difficult to give a necessary and sufficient condition in a finitely computable manner (like using only vertex-matrix information, where vertex matrices are those matrices formed at the vertices of the interval parameters), but after intense research of many years, it was only recently that a method was presented that gives a necessary and sufficient vertex solution for checking the robust stability of a linear interval parameter matrix family [24]. All these techniques involve considerable computation to arrive at the robust stability bounds, but these techniques never delved into the sign pattern of the elements of the matrix and thus never exploited this sign structure. But now with the ecological-sign-stability concept, it is clear that by paying attention to the sign pattern of the given matrix-element variations, much more can be said about the robust stability of the perturbed matrices. This aspect is elaborated as follows.

3. *Robust Stability Analysis of a Class of Interval Matrices*

Consider the interval matrix family, in which each individual element varies independently within a given interval. Thus the interval matrix family is denoted by $A \in [A^L, A^U]$ as the set of all matrices A that satisfy $(A^L)_{ij} \leq A_{ij} \leq (A^U)_{ij}$ for every i and j.

Now, consider a special class of interval matrix family in which, for each element that is varying, the lower bound, i.e., $(A^L)_{ij}$, and the upper bound, i.e., $(A^U)_{ij}$, are of the same sign. For example, consider the interval matrix given by

$$A = \begin{bmatrix} 0 & a_{12} & a_{13} \\ a_{21} & 0 & 0 \\ a_{31} & 0 & a_{33} \end{bmatrix}$$

with the elements a_{12}, a_{13}, a_{21}, a_{31}, and a_{33} being uncertain varying in some given intervals as follows:

$$2 \leq a_{12} \leq 5$$
$$1 \leq a_{13} \leq 4$$
$$-3 \leq a_{21} \leq -1$$
$$-4 \leq a_{31} \leq -2$$
$$-5 \leq a_{33} \leq -0.5$$

4. *Qualitative Stability as a Sufficient Condition for Robust Stability of a Class of Interval Matrices: A Link Between Life Sciences and Engineering Sciences*

It is clear that ecological sign-stable matrices have the interesting feature that once the sign pattern is a sign-stable pattern, the stability of the matrix is independent of the magnitudes of the elements of the matrix. That this property has direct link to stability robustness of matrices with structured uncertainty was recognized in earlier papers on this topic [18,19].

In [25], a viewpoint was put forth that advocates using the qualitative-stability concept as a means of achieving robust stability in the standard uncertain-matrix theory. Then, a sufficient condition for checking the robust stability of a class of interval matrices was presented using the qualitative-stability concept. This argument is illustrated with the following examples:

Example 3: consider the above given interval matrix. Once it is recognized that the signs of the interval entries in the matrix are not changing (within the given intervals), the sign matrix can be formed. The sign matrix for this interval matrix is given by

$$A = \begin{bmatrix} 0 & + & + \\ - & 0 & 0 \\ - & 0 & - \end{bmatrix}$$

The above sign matrix is known to be qualitative (sign) stable. Since sign stability is independent of magnitudes of the entries of the matrix, it can be concluded that the above interval matrix is robustly stable in the given interval ranges. If the robust stability of this interval matrix is to be ascertained by the methods of robustness theory of mathematical sciences, one needs to resort to the extreme-point solution offered in [24], which would have been computationally expensive because it involves checking the Hurwitz stability of the $2^5 = 32$ vertex matrices first and then following the algorithm to check the virtual stability of the 32 KN (Kronecker Nonsingularity) matrices in the higher-dimensional Kronecker–Lyapunov matrix space. But, in the above matrix, once it is realized that the sign of the matrix entries is not changing within the given intervals, the qualitative-stability concept can readily be applied and it can be concluded that the above interval matrix is robustly stable, because with only signs replacing the entries it is observed that the above matrix is Hurwitz stable irrespective of the magnitudes of those entries. Thus the robust stability of the entire interval-matrix family is established without resorting to any algorithms related to robust-stability literature. Incidentally, if the vertex algorithm of [24] is applied for this problem, it can be also concluded that this interval-matrix family is indeed Hurwitz stable in the given interval ranges.

In fact, more can be said about the robust stability of this matrix family using the sign-stability application. This matrix family is indeed robustly stable, not only for those given interval ranges above, but also for any large interval ranges in those elements as long as those interval ranges are such that the elements do not change signs in those interval ranges. Thus, elements a_{12} and a_{13} can vary along the entire positive real line, and elements a_{21}, a_{31}, and a_{33} can vary along the entire negative real line simultaneously, and still the resulting matrices are all stable. In other words, if this matrix were the plant matrix for a linear state-space system, that particular linear system has infinite bounds for robust stability in the specific sign preserving variations in the elements of that matrix. It could not have been possible to conclude this but for the usefulness of the sign-stability concept.

In the above discussion, the emphasis was on exploiting the sign pattern of a matrix in robust-stability analysis of matrices. Thus, the tolerable perturbations are direction-sensitive. Also, no perturbation is allowed in the structural zeros of the ecological sign-stable matrices. In what follows, it is shown that ecological sign-stable matrices can still possess superior robustness properties even under norm-bounded perturbations, in which perturbations in structural zeros are also allowed in ecological sign-stable matrices.

Toward this objective, the stability-robustness measures of linear state-space systems as discussed in [22,23] are considered. In other words, a linear state-space plant matrix A, which is assumed to be Hurwitz stable, is considered. Then, assuming a perturbation matrix E in the A matrix, the question as to how much of norm of the perturbation matrix E can be tolerated to maintain stability is asked. Note that in this norm-bounded perturbation discussion, the elements of the perturbation matrix can vary in various directions without any restrictions on the signs of the elements of that matrix. Even though there are many stability-robustness measures discussed in [23], in what follows, focus is placed on a specific robustness measure that makes use of the condition-number concept. Then, the advantage of sign-stable matrices over regular Hurwitz-stable matrices with respect to this particular robustness measure is explored.

5. *Comparison of Robustness Index of Different Matrices*

Consider a given Hurwitz-stable matrix A_0 with perturbation E such that

$$A = A_0 + E \tag{7}$$

where A is any one of the perturbed matrices.

A sufficient bound μ for the stability of the perturbed system is given on the spectral norm of the perturbation matrix as

$$\sigma_{\max}(E) < \frac{|\mathrm{Re}(\lambda(A_0))|_{\min}}{\kappa} = \mu \tag{8}$$

where κ is the condition number of the modal matrix of A_0.

A theorem which clearly establishes the superiority of ecological sign-stable matrices over other nonecological sign-stable matrices with respect to the above stability-robustness measure is now stated.

Theorem 3: consider a sign-stable matrix A and a non-sign-stable Hurwitz-stable matrix B where $a_{ii} = b_{ii} = c < 0 \quad \forall\, i$. The interactions in A are such that the elements in a p–p link have equal magnitude and the elements of B have identical corresponding magnitudes as that of A in the nonzero entries such that B is normal ($B^T B = BB^T$). This means that the interactions in B are not necessarily predator–prey in nature. Let these normal matrices be denoted as A_N and B_N. In addition, A_N and B_N are normalized and denoted by A_{NN} and B_{NN} where $A_{NN} = \frac{A_N}{\sigma_{\max}(A_N)}$ and $B_{NN} = \frac{B_N}{\sigma_{\max}(B_N)}$ so that $\sigma_{\max}(A_{NN}) = \sigma_{\max}(B_{NN}) = 1$. For above matrices A_{NN} and B_{NN},

$$|\mathrm{Re}(\lambda(A_{NN}))|_{\min} > |\mathrm{Re}(\lambda(B_{NN}))|_{\min} \tag{9a}$$

i.e.,

$$\mu(A_{NN}) > \mu(B_{NN}) \tag{9b}$$

In other words, a unit-norm, normal ecological sign-stable matrix is more robust that a unit-norm, normal nonecological sign-stable Hurwitz-stable matrix.

Proof: consider any square matrix H with all equal negative diagonal elements. This matrix can be written as

$$H = [H]_{\mathrm{diag}} + [H]_{\mathrm{int}} = cI + [H]_{\mathrm{int}}$$

where H_{int} is the matrix with all off-diagonal elements. Hence all the diagonal elements of H_{int} are 0. H_{int} can now be written as

$$H_{\mathrm{int}} = (H_{\mathrm{int}})_s + (H_{\mathrm{int}})_{sk}$$

where $(\cdot)_s$ is the symmetric part of the matrix and $(\cdot)_s$ and $(\cdot)_{sk}$ is the skew-symmetric part of the matrix.

Thus, $H = cI + [H_{\mathrm{int}}]_s + [H_{\mathrm{int}}]_{sk} = H_s + H_{sk}$ where

$$H_s = cI + [H_{\mathrm{int}}]_s \quad \text{and} \quad H_{sk} = [H_{\mathrm{int}}]_{sk}$$

From above, it is observed that

$$\lambda_i[H_s] = \lambda_i[cI + (H_{\mathrm{int}})_s] = c + \lambda_i[(H_{\mathrm{int}})_s]$$

Let A and B be the matrices as described in the theorem. Then,

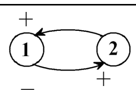
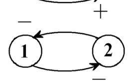
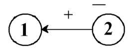
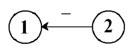
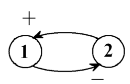
$$[A_{\mathrm{int}}]_s \equiv 0$$

$[B_{\mathrm{int}}]_s \neq 0$ (because of non-predator–prey interactions).

Note that $[B_{\mathrm{int}}]_s$ has zeros on its diagonal; hence, its real eigenvalues cannot all be negative and cannot all be positive because they have to add up to zero to satisfy the trace condition.

That means some eigenvalues of $[B_{\mathrm{int}}]_s$ are negative and some are positive. Therefore,

Table 1 Types of interactions between two species in an ecosystem.

Interaction type	Digraph representation	Matrix representation
Mutualism		$\begin{bmatrix} * & + \\ + & * \end{bmatrix}$
Competition		$\begin{bmatrix} * & - \\ - & * \end{bmatrix}$
Commensalism		$\begin{bmatrix} * & + \\ 0 & * \end{bmatrix}$
Ammensalism		$\begin{bmatrix} * & - \\ 0 & * \end{bmatrix}$
Predation (parasitism)		$\begin{bmatrix} * & + \\ - & * \end{bmatrix}$

$$|\lambda(B_{NNs})|_{\min} < |\lambda(A_{NNs})|_{\min}$$

Since all the matrices are normal, $|\mathrm{Re}(\lambda(A_{NN}))|_{\min} = |(\lambda(A_{NNs}))|_{\min}$ and $|\mathrm{Re}(\lambda(B_{NN}))|_{\min} = |(\lambda(B_{NNs}))|_{\min}$.

Therefore it can be seen that $|\mathrm{Re}(\lambda(A_{NN}))|_{\min} > |\mathrm{Re}(\lambda(B_{NN}))|_{\min}$ Q.E.D.

Example 4: This theorem is now illustrated by the following example, shown in Table 2, where one normalized sign-stable, pure-predator–prey-link-structure matrix with identical self-regulatory intensities and equal interaction strengths is compared with two different types of normalized, non-sign-stable, non-predator–prey-interaction, Hurwitz-stable normal matrices.

B_{N1} is a symmetric matrix (which in turn is a normal matrix). B_{N2} is a nonsymmetric, Hurwitz-stable, non-sign-stable, non-predator–prey-link matrix that is normal. The corresponding magnitudes of all the nonzero entries in all the three matrices are identical. With the magnitudes of the elements of all the matrices being the same, the effect of the sign pattern on the stability and robustness of the matrix is highlighted.

Example 5: Having established the (superior) robustness characteristics of a sign-stable matrix with pure predator–prey links, it is of interest to compare the robustness characteristics of this sign-stable matrix with a Hurwitz matrix with all predator–prey links that is normal but not sign stable anymore. This comparison is illustrated in Table 3.

From this it can easily be concluded that in matrices with pure-predator–prey-link structure, imposing the condition of sign stability pushes the eigenvalues further left and thus improves the robustness bounds. This can easily be proved mathematically.

Hence pure-predator–prey-link matrices are seen to be superior to non-predator–prey-link matrices from a stability-robustness point of view. And within this set of predator–prey-link matrices, sign-stable matrices are even better. From the above analysis it is clear that sign-stable matrices possess superior robustness qualities. This gives impetus to design controllers that drive the closed-loop system to an ecological sign-stable pattern. Toward this objective, an algorithm for the design of a controller based on concepts from ecological sign stability is now presented.

IV. Robust Control Design Using an Ecological-Sign-Stability Approach

Now that the eigenvalue distribution, condition number (normality), and robust-stability properties of ecological sign-stable matrices are studied, it is proposed to use these results to design robust controllers for engineering systems. In this paper, we focus on designing full-state-feedback controllers, assuming all states are available for measurement. Specifically, as discussed in [26], consider the linear state-space model given by

$$\dot{x} = Ax + Bu$$

where x is the state-variable vector and u is the control variable.

It is proposed to design a full-state-feedback controller so that the closed-loop-system matrix is given by

$$A_{n\times n} + B_{n\times m}G_{m\times n} = A_{\mathrm{cl}n\times n} \tag{11}$$

where the dimensions of the matrices are highlighted anticipating their use in the algorithm to be discussed later. From the above,

$$BG = A_{\mathrm{cl}} - A = A_a \tag{12}$$

The desired closed-loop matrix is thought of as the target closed-loop matrix. In this design method, it is intended to select this target closed-loop-system matrix A_{cl} from ecological principles, namely that there be predation–prey links (off-diagonal elements) and as many self-regulatory species as possible (as many negative diagonal elements as possible). Thus, armed with the justification given in the previous section, the target closed-loop-system matrix A_{cl} is taken to be a quantitative sign-stable matrix, wherein the magnitudes of the sign-stable matrix entries can be treated as design variables, trying to be as close to the desirable properties (discussed in the previous section) as possible.

Now, consider the following cases, based on the structure of the B matrix.

Case 1: B is a square, nonsingular matrix. Then the control-gain matrix G is given by

$$G = B^{-1}A_a \tag{13}$$

Case 2: B is a nonsquare matrix. Arranging all the elements of the above matrices in vector form, the above matrix equation can be rewritten as

$$\tilde{B}_{n^2\times mn}\underline{g}_{mn\times 1} = \underline{a}_{\mathrm{cl}} - \underline{a} = \underline{a}_{a\,n^2\times 1} \tag{14}$$

Symbolically, the terms in this equation can be expressed as follows:

Table 2 Comparison of robustness index of B_{N1}, B_{N2}, and A_N matrices.

Matrix type		$(\cdot)_N$	$(\cdot)_{NN}(=\frac{(\cdot)_N}{\sigma_{\max}((\cdot)_N)})$	$\lambda_i[(\cdot)_{NN}]$	μ
Symmetric matrix	B_{N1}	$\begin{bmatrix} -2 & -1 & 0 \\ -1 & -2 & -1 \\ 0 & -1 & -2 \end{bmatrix}$	$\begin{bmatrix} -0.5858 & -0.2928 & 0 \\ -0.2928 & -0.5858 & -0.2928 \\ 0 & -0.2928 & -0.5858 \end{bmatrix}$	$\begin{bmatrix} -1.0 \\ -0.5858 \\ -0.1716 \end{bmatrix}$	0.1716
Non- symmetric, no predator–prey links, normal matrix	B_{N2}	$\begin{bmatrix} -2 & -1 & 0 \\ 0 & -2 & -1 \\ -1 & 0 & -2 \end{bmatrix}$	$\begin{bmatrix} -0.6667 & -0.3333 & 0 \\ 0 & -0.6667 & -0.3333 \\ -0.3333 & 0 & -0.6667 \end{bmatrix}$	$\begin{bmatrix} -1.0 \\ -0.5 + j0.2887 \\ -0.5 - j0.2887 \end{bmatrix}$	0.5
Pure-predator–prey-link sign-stable matrix with equal interaction intensities and identical self-regulation rates.	A_N	$\begin{bmatrix} -2 & -1 & 0 \\ 1 & -2 & -1 \\ 0 & 1 & -2 \end{bmatrix}$	$\begin{bmatrix} -0.8165 & -0.4082 & 0 \\ 0.4082 & -0.8165 & -0.4082 \\ 0 & 0.4082 & -0.8165 \end{bmatrix}$	$\begin{bmatrix} -0.8165 \\ -0.8165 + j0.5774 \\ -0.8165 - j0.5774 \end{bmatrix}$	0.8165

$$\tilde{B} = \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \\ \vdots \\ \tilde{B}_n \end{bmatrix}_{n^2\times mn} \quad \text{where } \tilde{B}_1 = \begin{bmatrix} B_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & B_1 \end{bmatrix}_{n\times mn}$$

$$\tilde{B}_2 = \begin{bmatrix} B_2 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & B_2 \end{bmatrix}_{n\times mn} \cdots \tilde{B}_n = \begin{bmatrix} B_n & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & B_n \end{bmatrix}_{n\times mn} \tag{15a}$$

and B_i $(i = 1, \ldots, n)$ is the ith row of B:

$$\underline{g} = \begin{bmatrix} \tilde{G}_1 \\ \tilde{G}_2 \\ \vdots \\ \tilde{G}_n \end{bmatrix} \quad \text{where } \tilde{G}_i \text{ is the } i\text{th column of } G \tag{15b}$$

$$\underline{a} = \begin{bmatrix} \tilde{A}_1^T \\ \tilde{A}_2^T \\ \vdots \\ \tilde{A}_n^T \end{bmatrix} \quad \text{where } \tilde{A}_i \text{ is the } i\text{th row of } A \tag{15c}$$

Similarly,

$$\underline{a}_{\text{cl}} = \begin{bmatrix} A_{\text{cl}_1} \\ A_{\text{cl}_2} \\ \vdots \\ A_{\text{cl}_n} \end{bmatrix} \tag{15d}$$

and

$$\underline{a}_a = \begin{bmatrix} A_{a_1} \\ A_{a_2} \\ \vdots \\ A_{a_n} \end{bmatrix} \tag{15e}$$

$\underline{g}$ is a vector formed by concatenating successive columns of matrix G. $\underline{a}$ is a vector formed by concatenating successive rows of matrix A in the form of a column vector. Similarly, vectors $\underline{a}_{\text{cl}}$ and $\underline{a}_a$ are formed from matrices A_{cl} and A_a, respectively. The above notation is illustrated in the following example.

Example 6: let $BG = A_a (=A_{\text{cl}} - A)$ where

$$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, \quad G = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}, \quad A_a = \begin{bmatrix} a_{a11} & a_{a12} \\ a_{a21} & a_{a22} \end{bmatrix}$$

Using Eqs. (15) the following linear system is formed:

$$\begin{bmatrix} b_{11} & b_{12} & 0 & 0 \\ 0 & 0 & b_{11} & b_{12} \\ b_{21} & b_{22} & 0 & 0 \\ 0 & 0 & b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} g_{11} \\ g_{21} \\ g_{12} \\ g_{22} \end{bmatrix} = \begin{bmatrix} a_{a11} \\ a_{a12} \\ a_{a21} \\ a_{a22} \end{bmatrix}$$

The control-gain vector $\underline{g}_{mn\times 1}$ is obtained by applying the conditions for existence of a solution for a system of linear equations as discussed in [27]:

1. If $\text{rank}(\tilde{B}) \neq \text{rank}(\tilde{B}|\underline{a}_a)$, no solution exists for the matrix G (overdetermined system).

2a. If $\text{rank}(\tilde{B}) = \text{rank}(\tilde{B}|\underline{a}_a) = mn$ (number of elements of matrix G), then a unique solution exists for a particular set of $\underline{a}_a$ elements.

2b. If $\text{rank}(\tilde{B}) = \text{rank}(\tilde{B}|\underline{a}_a) < mn$ (number of elements of matrix G), then infinite solutions exist for a particular set of magnitudes for $\underline{a}_a$ elements (underdetermined system).

From the above algorithm, it can be seen that the control-gain determination heavily depends on the numerical values taken for the sign-stable closed-loop-system matrix as well as the structures of the open-loop matrices A and B.

The proposed algorithm is illustrated with two applications in the Aerospace field: satellite attitude control problem and aircraft lateral dynamics control problem.

A. Satellite Formation Flying Control Problem

Example 7: in [28], a control design for the satellite formation flying problem was discussed. For this system, in [19,20], a controller was designed using the concept of sign stability but with no formal procedure or justification for the resulting closed-loop-system matrix. The above control algorithm is now illustrated for the same example. Following is the satellite attitude dynamics and control problem discussed in [19,20]:

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{y} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2\omega \\ 0 & 3\omega^2 & -2\omega & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ y \\ \dot{y} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} T_x \\ T_y \end{bmatrix} \tag{16}$$

where x, $\dot{x}$, y, and $\dot{y}$ are the state variables, and T_x and T_y are the control variables.

For example, when $\omega = 1$, the system becomes

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 3 & -2 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

A controller is to be designed such that the closed-loop system is ecological sign stable with magnitudes decided by the analysis and the algorithm described in the previous sections.

Accordingly, an ecological sign-stable closed-loop system is chosen such that 1) the closed loop matrix has as many pure predator–prey links as possible; and 2) it also has as many negative diagonal elements as possible. Taking the above points into consideration, the following sign pattern is chosen, which is appropriate for the given A and B matrices:

$$A_{\text{cl}_{ss}} = \begin{bmatrix} 0 & 0 & + & 0 \\ 0 & 0 & 0 & + \\ - & 0 & - & + \\ 0 & - & - & - \end{bmatrix}$$

The magnitudes of the entries of the above sign matrix are decided by the stability robustness analysis theorem discussed above:

1) All nonzero a_{ii} are identical.

2) $a_{ij} = -a_{ji}$ for all nonzero a_{ij}, else $a_{ij} = a_{ji} = 0$.

The magnitudes of the entries, therefore, are

$$A_{\text{cl}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -1 & 2 \\ 0 & -1 & -2 & -1 \end{bmatrix}$$

Hence, all the pure predator–prey links are of equal interaction strengths, and the nonzero diagonal elements have identical self-regulatory intensities. Using the algorithm given above, the gain matrix is computed as shown below.

Since the first rows are not affected by the B matrix, the $\tilde{B}$ matrix can be reduced to a $mn \times mn$ matrix. So now there are eight equations for the eight unknowns (elements of the gain matrix). In the $\tilde{B}\,\underline{g} = \underline{a}_{\text{cl}} - \underline{a} (=\underline{a}_a)$ form, the system is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} g_{11} \\ g_{21} \\ g_{12} \\ g_{22} \\ g_{13} \\ g_{23} \\ g_{14} \\ g_{24} \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \\ -1 \\ -2 \\ 2 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 3 \\ 0 \\ -2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ -4 \\ -1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

From the algorithm,

$$G_{es} = \begin{bmatrix} -1.0 & 0 & -1.0 & 0 \\ 0 & -4.0 & 0 & -1.0 \end{bmatrix}$$

The closed-loop matrix $A_{\text{cl}}(=A + BG_{es})$ is sign-stable and hence can tolerate any amount of variation in the magnitudes of the elements with the sign pattern kept constant.

In this application, it is clear that all nonzero elements in the open-loop matrix (excluding elements A_{13} and A_{24} since they are connected to states resulting from the transformation of the system into a set of first-order differential equations) are functions of the angular velocity ω. Hence, real-life perturbations in this system occur only due to variation in angular velocity ω. Therefore, a perturbed satellite system is simply an A matrix generated by a different ω. This means that not every randomly chosen matrix represents a physically perturbed system and that, for practical purposes, stability of the matrices generated as mentioned above (by varying ω) is sufficient to establish the robustness of the closed-loop system. It is only because of the ecological perspective that these structural features of the system are brought to light. Also, it is the application of these ecological principles that makes the control design for satellite formation flying this simple and insightful.

To demonstrate the magnitude independence of stability of the closed-loop system, keeping the given B matrix and the above designed G_{es} (designed for $\omega = 1$) constant, time histories of the four states in each of the perturbed cases are plotted in Fig. 6.

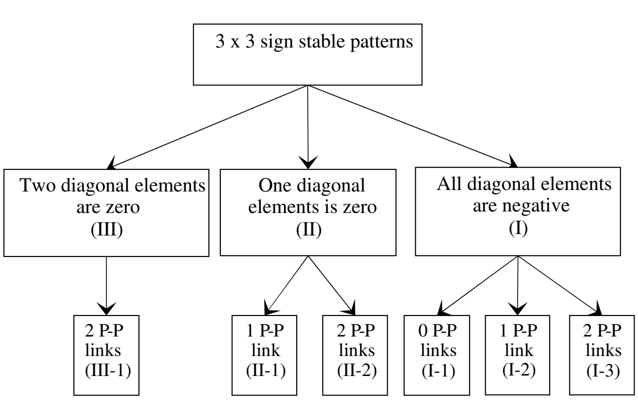


Fig. 5 Categorization of pure-predator–prey-link models.

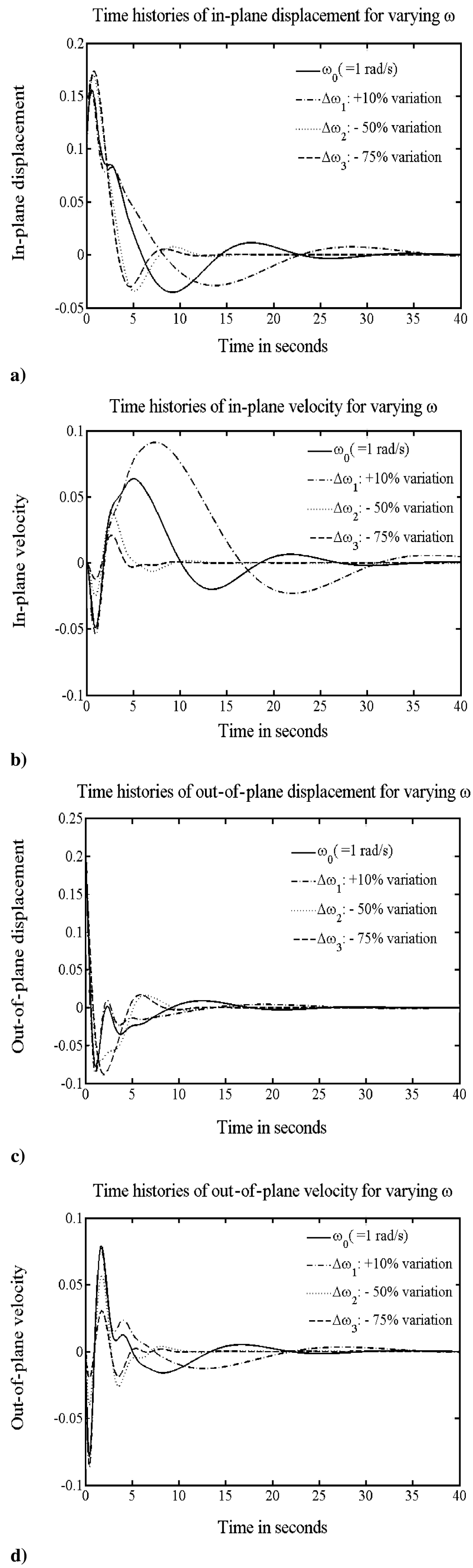


Fig. 6 Time histories of the states' a) in-plane displacement, b) in-plane velocity, c) out-of-plane displacement, and d) out-of-plane velocity for varying.

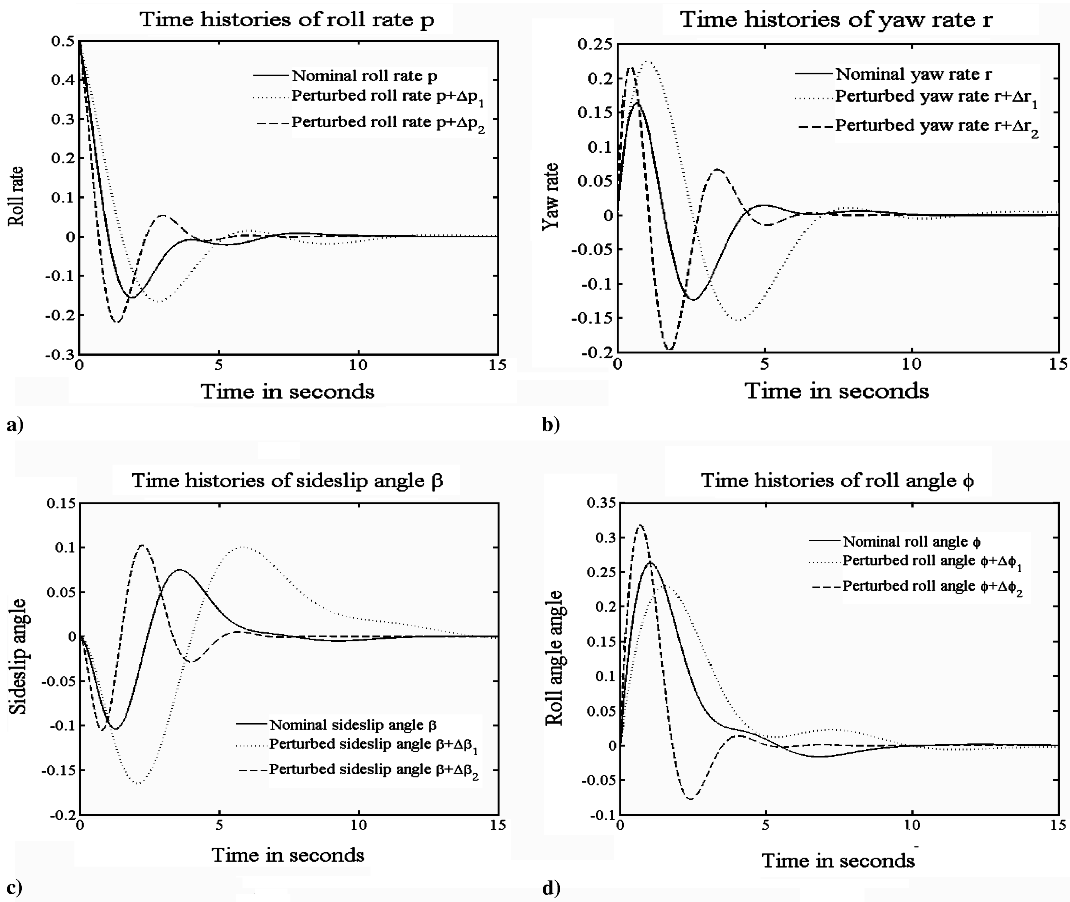


Fig. 7 Time histories of the states' a) roll rate, b) yaw rate, c) sideslip angle, and d) roll angle under perturbations.

As can be seen in the above plots, the control gain can be deemed to be a robust control gain as it produces stable closed-loop systems in the presence of perturbations.

B. Aircraft-Flight-Control Problem

Example 8: consider the problem of aircraft lateral dynamics from [26]. An approximate linear model of the lateral dynamics of an aircraft for a particular set of flight conditions is given by

$$\begin{bmatrix} \dot{p} \\ \dot{r} \\ \dot{\beta} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} -10 & 0 & -10 & 0 \\ 0 & -0.7 & 9 & 0 \\ 0 & -1 & -0.7 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p \\ r \\ \beta \\ \phi \end{bmatrix} + \begin{bmatrix} 20 & 28 \\ 0 & -3.13 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \delta_a \\ \delta_r \end{bmatrix} \tag{17}$$

where $x(t) \in R^4$ is the state vector consisting of the four state variables p (roll rate), r (yaw rate), β (sideslip angle), and ϕ (roll angle), and $u(t) \in R^2$ is the control vector consisting of the two control surface deflections δ_a (aileron angle) and δ_r (rudder angle).

As established in Section III, it is desired to have a pure predator–prey interaction closed loop matrix. The following is a desired closed loop sign pattern that can be achieved, given the structure of B.

$$\begin{bmatrix} - & - & 0 & - \\ + & - & + & 0 \\ 0 & - & - & 0 \\ + & 0 & 0 & 0 \end{bmatrix}$$

Considering the logic provided in Sec. III, the magnitudes are chosen such that the pure predator–prey interactions have equal magnitudes and the self-regulatory intensities are identical. Therefore, the corresponding quantitative closed-loop-system matrix is given by

$$A_{\text{cl}} = \begin{bmatrix} -0.7 & -1 & 0 & -1 \\ 1 & -0.7 & 1 & 0 \\ 0 & -1 & -0.7 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Table 3 Comparison of robustness index of A_N and A_{Npp}.

Matrix type		$(\cdot)_N$	$(\cdot)_{NN}(=\frac{(\cdot)_N}{\sigma_{\max}((\cdot)_N)})$	$\lambda_i[(\cdot)_{NN}]$	μ
Pure-predator–prey-link sign-stable matrix with equal interaction intensities and identical self-regulation rates.	A_N	$\begin{bmatrix} -2 & -1 & 0 \\ 1 & -2 & -1 \\ 0 & 1 & -2 \end{bmatrix}$	$\begin{bmatrix} -0.8165 & -0.4082 & 0 \\ 0.4082 & -0.8165 & -0.4082 \\ 0 & 0.4082 & -0.8165 \end{bmatrix}$	$\begin{bmatrix} -0.8165 \\ -0.8165 + j0.5774 \\ -0.8165 - j0.5774 \end{bmatrix}$	0.8165
All predator–prey links	A_{Npp}	$\begin{bmatrix} -2 & -1 & 1 \\ 1 & -2 & 1 \\ -1 & -1 & -2 \end{bmatrix}$	$\begin{bmatrix} -0.7559 & -0.3780 & 0.3780 \\ 0.3780 & -0.7559 & 0.3780 \\ -0.3780 & -0.3780 & -0.7559 \end{bmatrix}$	$\begin{bmatrix} -0.7559 \\ -0.7559 + j0.6547 \\ -0.7559 - j0.6547 \end{bmatrix}$	0.7559

Applying the algorithm with the above A_{cl}, the control gain is obtained as

$$G_{es} = \begin{bmatrix} 0.5097 & -0.05 & 0.1421 & -0.05 \\ -0.3194 & 0 & 2.5559 & 0 \end{bmatrix}$$

This closed-loop matrix, $A_{cl}(=A + BG_{es})$, therefore, tolerates perturbation of any magnitude as long as the sign pattern remains unchanged. The eigenvalues of the closed loop system are

$$-0.6152 + j1.5827 \qquad -0.4348 + j0.5725$$
$$-0.6152 - j1.5827 \qquad -0.4348 + j0.5725$$

which corresponds to a damping ratio of $\varsigma = 0.6048$ and to a natural frequency of $\omega_n = 0.7189$.

In what follows, the usefulness of the above controller as a robust controller is demonstrated by assuming various sign-preserving perturbations in the closed-loop-system matrix and the resulting time histories of states guaranteeing stability under these perturbations. For brevity, simulations are shown in Fig. 7 for two specific realizations of perturbations. These perturbations may be interpreted as variations in various stability derivatives that appear as entries in the state space matrices.

Consider

$$\text{Nominal } A_{cl} = \begin{bmatrix} -0.7 & -1 & 0 & -1 \\ 1 & -0.7 & 1 & 0 \\ 0 & -1 & -0.7 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{Perturbation 1 } \Delta A_{cl1} = \begin{bmatrix} 0.2 & 0.5 & 0 & 0.25 \\ -0.1 & 0.22 & -0.7 & 0 \\ 0 & 0.3 & 0.3 & 0 \\ -0.4 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{Perturbation 2 } \Delta A_{cl2} = \begin{bmatrix} -0.2 & -0.48 & 0 & -0.8 \\ 0.9 & -0.3 & 0.82 & 0 \\ 0 & -3.9 & -1 & 0 \\ 0.73 & 0 & 0 & 0 \end{bmatrix}$$

In both of the above examples the emphasis was on describing the control-design procedure based on ecological sign stability. It can be seen that there is considerable flexibility in deciding the magnitudes of the resulting sign-stable closed-loop-system matrix. In fact, in addition to magnitudes, even the sign-stable sign patterns may also be taken as design variables to achieve various control-design objectives. The ideas proposed in this paper offer concrete guidelines to determine the desired target closed-loop systems in methods for linear systems control design.

V. Conclusions

The main purpose of this paper is to highlight the ecological principles inherent in ecological-sign-stability phenomenon and bring out their implications in the matrix-theory properties encountered in engineering systems analysis and design. The intent is to be able to understand the inherent features of ecological systems and exploit these features in building a robust engineering system. Toward this objective, the results presented in this paper provide significant insights into the interrelationship between ecological principles and the corresponding matrix-theory implications. The first insight of interest (to both ecologists and engineers) is that, for any sign-stable matrix, it is shown that the bounds on the real parts of the eigenvalues are simply the diagonal elements of the matrix itself. To be able to establish the region of location of eigenvalues without resorting to any elaborate calculations or computations is a property that is exclusive to ecological sign-stable matrices. Similarly it is of significance to realize that if the strengths of interactions and self-regulation levels are all equal, the ecological sign-stable matrix with pure predator–prey interactions becomes a normal matrix, and hence the condition number of its modal matrix is always one. Using this result, it is proved that such matrices are always more robust from a real-parameter-variations point of view compared with other general matrices. Thus, an interesting link between sign stability and robust stability is provided in this paper. Finally, based on these results, a new control-design method involving sign stability is presented, and its usefulness is demonstrated with the help of two important applications in the aerospace field. The results presented in this paper can assist in the use of ecological system principles to build highly robust engineering systems.

Appendix A

Proof of Theorem 2: condition for normality of a 2×2 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is

$$a_{11} = a_{22}, \qquad a_{12}^2 = a_{21}^2 (\Rightarrow a_{21} = a_{12} \quad \text{or} \quad a_{21} = -a_{12})$$

For example, the 2×2 predator–prey model

$$\begin{bmatrix} -x & y \\ -y & -x \end{bmatrix}$$

satisfies the above condition. Therefore, with the given condition on the diagonal elements, a single predator–prey link with identical self-regulation rates and equal interaction strengths is always normal.

Extension to higher order systems: consider the 3×3 matrix with pure-predator–prey-link structure

$$A = \begin{bmatrix} a & a_{12} & a_{13} \\ -a_{12} & a & a_{23} \\ -a_{13} & -a_{23} & a \end{bmatrix}$$

A can be partitioned as

$$\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$$

where

$$A_1 = \begin{bmatrix} a & a_{12} \\ -a_{12} & a \end{bmatrix}, \quad A_2 = \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix}, \quad A_3 = [-a_{13} \;\; -a_{23}], \quad A_4 = [a]$$

Here, A_1 and A_4 are normal matrices, and $A_3 = -A_2^T$ (due to predator–prey interaction). For A to be normal, $A^T A = A A^T$:

$$\Rightarrow \begin{bmatrix} A_1^T & A_3^T \\ A_2^T & A_4^T \end{bmatrix} \cdot \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} A_1^T & A_3^T \\ A_2^T & A_4^T \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} A_1^T A_1 + A_3^T A_3 & A_1^T A_2 + A_3^T A_4 \\ A_2^T A_1 + A_4^T A_3 & A_2^T A_2 + A_4^T A_4 \end{bmatrix}$$

$$= \begin{bmatrix} A_1 A_1^T + A_2 A_2^T & A_1 A_3^T + A_2 A_4^T \\ A_3 A_1^T + A_4 A_2^T & A_3 A_3^T + A_4 A_4^T \end{bmatrix}$$

Since A_1 and A_4 (here A_4 is a scalar quantity) are normal matrices and $A_3 = -A_2^T$, the above equation becomes

$$\begin{bmatrix} A_1^T A_1 + A_2 A_2^T & A_1^T A_2 - A_2 A_4 \\ A_2^T A_1 - A_4^T A_2^T & A_2^T A_2 + A_4^T A_4 \end{bmatrix}$$

$$= \begin{bmatrix} A_1 A_1^T + A_2 A_2^T & -A_1 A_2 + A_2 A_4^T \\ -A_2^T A_1^T + A_4 A_2^T & A_2^T A_2 + A_4 A_4^T \end{bmatrix}$$

In each of the above matrices, the submatrix $(\cdot)_{21} = (\cdot)_{12}^T$ (where the fact that A_4 is used). Hence the property of symmetry is satisfied. For both the matrices to be equal,

$$A_1^T A_2 - A_2 A_4 = -A_1 A_2 + A_2 A_4^T \Rightarrow (A_1^T + A_1) A_2 = A_2 (A_4^T + A_4)$$

By virtue of the equal-strength predator–prey interactions, $(A_1^T + A_1)$ and $(A_4^T + A_4)$ are diagonal matrices with identical diagonal elements.

Hence, the equality above becomes

$$\text{diag}\,[2a, 2a].A_2 = A_2[2a]$$

which is true. Therefore, a 3×3 matrix with pure predator–prey interactions of equal strengths is always normal.

Similarly, a 4×4 equal-strength pure-predator–prey-link interactions matrix M can be written as

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

such that A and D are normal and of order two. Then,

$$P = M^T M = \begin{bmatrix} A^T & C^T \\ B^T & D^T \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A^T A + C^T C & A^T B + C^T D \\ B^T A + D^T C & B^T B + D^T D \end{bmatrix}$$

$$Q = M M^T = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} A^T & C^T \\ B^T & D^T \end{bmatrix} = \begin{bmatrix} AA^T + BB^T & AC^T + BD^T \\ CA^T + DB^T & CC^T + DD^T \end{bmatrix}$$

To establish the property of normality $P = Q$ must hold.

Since $A^T A = AA^T$, $D^T D = DD^T$, and $C = -B^T$,

$$\begin{bmatrix} A^T A + BB^T & A^T B - BD \\ B^T A - D^T B^T & B^T B + D^T D \end{bmatrix} = \begin{bmatrix} AA^T + BB^T & -AB + BD^T \\ -B^T A^T + DB^T & B^T B + DD^T \end{bmatrix}$$

Therefore, $P(1,1) = Q(1,1)$ and $P(2,2) = Q(2,2)$.

The other condition on the blocks of P and Q matrices for normality of M is

$$P_{(2,1)} = P_{(1,2)}^T = Q_{(2,1)} = Q_{(1,2)}^T$$

$$B^T A - D^T B^T = (A^T B - BD)^T \Rightarrow P_{(2,1)} = P_{(1,2)}^T - B^T A^T + DB^T = (-AB + BD^T)^T \Rightarrow Q_{(2,1)} = Q_{(1,2)}^T$$

The condition is satisfied if

$$P_{(1,2)} = Q_{(1,2)} \Rightarrow A^T B - BD = -AB + BD^T \Rightarrow A^T B + AB = BD^T + BD \Rightarrow (A^T + A)B = B(D^T + D)$$

Because of the equal-strength predator–prey interactions,

$$(A^T + A) = (D^T + D) = a_{ii}.I_{2\times 2}$$

Since identity matrices commute with every matrix, the equality $(A^T + A)B = B(D^T + D)$ holds.

Hence $P = Q$, and, therefore, the matrix M is always normal.

By induction, this proof can be extended to any $n \times n$ pure-predator–prey-link matrix that satisfies the conditions of identical self-regulation rates and equal interaction strengths. It is also observed that this property is imparted by the predator–prey interactions only. That is, the matrix need not be sign stable in order to be normal.

By the above analysis, an interesting property of predator–prey links is brought to light.

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References

[1] Edelstein-Keshet, L., *Mathematical Models in Biology*, McGraw Hill, New York, 1988.

[2] Hofbauer, J., and Sigmund, K., "Growth Rates and Ecological Models: ABC on ODE," *The Theory of Evolutions and Dynamical Systems*, Cambridge Univ. Press, London, 1988, pp. 29–59.

[3] Olek, S., "An Accurate Solution to the Multispecies Lotka–Volterra Equations," *SIAM Review*, Vol. 36, No. 3, 1994, pp. 480–488.
doi:10.1137/1036104

[4] May, R. M., and Leonard, W. J., "Nonlinear Aspects of Competition between Three Species," *SIAM Journal on Applied Mathematics*, Vol. 29, No. 2, 1975, pp. 243–253.
doi:10.1137/0129022

[5] Batiha, K., "Numerical Solutions of the Multispecies Predator–Prey Model by Variational Iteration Method," *Journal of Computer Science and Technology (English Language Edition)*, Vol. 3, No. 7, 2007, pp. 523–527.
doi:10.1016/j.camwa.2006.12.058

[6] Lee, C. S., and Leitmann, G., "On Optimal Long-Term Management of Some Ecological Systems Subject to Uncertain Disturbances," *International Journal of Systems Science*, Vol. 14, No. 8, 1983, pp. 979–994.
doi:10.1080/00207728308926509

[7] Corless, M., and Leitmann, G., "Adaptive Long-Term Management of Some Ecological Systems Subject to Uncertain Disturbances," *Optimal Control Theory and Economic Analysis 2, Second Viennese Workshop on Economic Applications of Control Theory*, Vienna, May 1984, pp. 3–29.
doi:10.1016/0096-3003(94)00118-N

[8] Neubert, M. G., and Caswell, C., "Alternatives to Resilience for Measuring the Responses of Ecological Systems to Perturbations," *Ecology*, Vol. 78, No. 3, 1997, pp. 653–665.

[9] Verdy, A., and Caswell, H., "Sensitivity Analysis of Reactive Ecological Dynamics," *Bulletin of Mathematical Biology*, Vol. 70, Aug. 2008, pp. 1634–1659.
doi:10.1007/s11538-008-9312-7

[10] Haddad, W. M., and Chellaboina, V., "Stability and Dissipativity Theory for Nonnegative Dynamical Systems: A Unified Analysis Framework for Biological and Physiological Systems," *Nonlinear Analysis: Real World Applications*, Vol. 6, No. 1, 2005, pp. 35–65.
doi:10.1109/.2001.980141

[11] Sandberg, W., "On the Mathematical Foundation of Compartmental Analysis in Biology, Medicine and Ecology," *IEEE Transactions on Circuits and Systems*, Vol. 25, No. 5, 1978, pp. 273–279.
doi:10.1109/TCS.1978.1084473

[12] Dambacher, J. M., Luh, H-K., Li, H. W., and Rossignol, P. A., "Qualitative Stability and Ambiguity in Model Ecosystems," *The American Naturalist*, Vol. 161, No. 6, 2003, pp. 876–888.
doi:10.1086/367590

[13] Logofet, D., *Matrices and Graphs: Stability Problems in Mathematical Ecology*, CRC Press, Boca Raton, FL, 1992, pp. 1–35.

[14] Kaszkurewicz, E., and Bhaya, A., *Matrix Diagonal Stability in Systems and Computation*, Birkhauser, Boston, MA, 2000, pp. 52–75.

[15] Quirk, J., and Ruppert. R., "Qualitative Economics and the Stability of Equilibrium," *Reviews in Economic Studies*, Vol. 32, No. 4, 1965, pp. 311–326.

[16] May. R., *Stability and Complexity in Model Ecosystems*, Princeton Univ. Press, Princeton, NJ, 1973, pp. 16–34.

[17] Jeffries, C., "Qualitative Stability and Digraphs in Model Ecosystems," *Ecology*, Vol. 55, No. 6, 1974, pp. 1415–1419.
doi:10.2307/1935470

[18] Yedavalli, R. K., "Qualitative Stability Concept from Ecology and its Use in the Robust Control of Engineering Systems," *Proceedings of the American Control Conference*, Minneapolis, MN, June 2006, pp. 5097–5102.
doi:10.1109/ACC.2006.1657531

[19] Yedavalli, R. K., "New Robust and Non-Fragile Control Design Method for Linear Uncertain Systems Using Ecological Sign Stability Approach," *Proceedings of the International Conference on Advances in Control and Optimisation of Dynamical Systems. (ACODS07)*, Bangalore, India, Feb. 2007, pp. 267–273.

[20] Yedavalli, R. K., "Robust Control Design for Linear Systems Using an Ecological Sign Stability Approach," *AIAA Journal of Guidance, Control and Dynamics*, Vol. 32, No. 1, 2009, pp. 348–352. doi:10.2514/1.35763

[21] Bernstein, D. S., "Matrix Classes and Transformations," *Matrix Mathematics: Theory, Facts and Formulas with Application to Linear Systems Theory*, Princeton Univ. Press, Princeton, NJ, 2005.

[22] Yedavalli, R. K., "Flight Control Application of New Stability Robustness Bounds for Linear Uncertain Systems," *Journal of Guidance, Control and Dynamics*, Vol. 16, No. 6, 1993. doi:10.2514/3.21124

[23] Dorato, P., and Yedavalli, R. K. (eds.), *Recent Advances in Robust Control*, IEEE Press, Piscataway, NJ, 1991, pp. 109–111.

[24] Yedavalli, R. K., "Robust Stability of Linear Interval Parameter Matrix Family Problem Revisited with Accurate Representation and Solution," *Proceedings of American Automatic Control Conference*, St. Louis, MO, June, 2009, pp. 3710–3717.

[25] Yedavalli, R. K., "Qualitative Stability Concept From Ecology and its Use in the Robust Stability Analysis of Linear Interval Parameter Systems," *Proceedings of the 5th IFAC Symposium on Robust Control Design*, Toulouse, France, July 2006.

[26] Yedavalli, R. K., and Devarakonda, N., "Ecological Sign Stability and its Use in Robust Control Design for Aerospace Applications," *Proceedings of IEEE Conference on Control Applications*, San Antonio, TX, Sept. 2008, pp. 912–917. doi:10.1109/CCA.2008.4629683

[27] Brogan, W. L., "Simultaneous Linear Algebraic Equations," *Modern Control Theory*, 2nd ed., Prentice Hall, Englewood Cliffs, NJ, 1985, pp. 125–131.

[28] Yedavalli, R. K., and Sparks. A., "Satellite Formation Flying and Control Design Based on Hybrid Control System Stability Analysis," *Proceedings of the American Control Conference*, Chicago, IL, June 2000, pp. 2210–2214. doi:10.1109/ACC.2000.879593